XII-MATHEMATICS: VECTOR ALGEBRA II AND APPLICATIONS

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PRE-REQUISITES

- Determinants
- Algebra of vectors
- Position vector of a point
- Cartesian coordinates
- Units vectors $\vec{i}, \vec{j}, \vec{k}$
- Dot product of two vectors
- Cross product of two vectors

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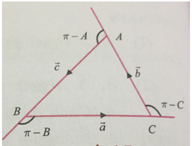
Outcomes

- Upon completion of this chapter, students will be able to
  - apply scalar and vector products of two and three vectors
  - solve problems in geometry, trigonometry and physics
  - derive equations of a line in parametric, non-parametric and cartesian forms in different situations
  - derive equations of a plane in parametric, non-parametric and cartesian forms in different situations
  - find, angle between the lines, and distance between skew lines
  - find the coordinates of the image of a point

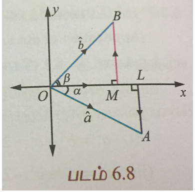
- Example 6.1 (Cosine formulae) With usual notation, in a triangle ABC, prove the following by vector method,
  - (i) $a^2 = b^2 + c^2 - 2bc\cos A$
  - (ii) $b^2 = c^2 + a^2 - 2ca\cos B$
  - (iii) $c^2 = a^2 + b^2 - 2ab\cos C$
- Solution:



படம் 6.7

- $\vec{BC} + \vec{CA} + \vec{AB} = \vec{0}$
- $\implies \vec{BC} = -\vec{CA} - \vec{AB}$
- $\implies a^2 = |\vec{BC}|^2 = \vec{BC} \cdot \vec{BC} = (-\vec{CA} - \vec{AB}) \cdot (-\vec{CA} - \vec{AB})$
- $\implies a^2 = \vec{CA} \cdot \vec{CA} + \vec{CA} \cdot \vec{AB} + \vec{AB} \cdot \vec{CA} + \vec{AB} \cdot \vec{AB}$
- $\implies a^2 = |\vec{CA}|^2 + |\vec{AB}|^2 + 2\vec{CA} \cdot \vec{AB}$
- $\implies a^2 = b^2 + c^2 + 2bc\cos(\pi - A)$
- $\implies a^2 = b^2 + c^2 - 2bc\cos A$

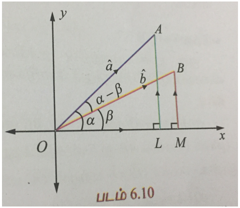
- Exercise 6.1(9) Prove the following by vector method,
  - (i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
  - (ii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- Solution:



$$\hat{a} = \cos\alpha\hat{i} - \sin\alpha\hat{j} \ \hat{b} = \cos\beta\hat{i} + \sin\beta\hat{j}$$

- $\hat{a}\cdot\hat{b} = |\hat{a}||\hat{b}|\cos(\alpha+\beta)$
- $\implies (\cos\alpha\hat{i} - \sin\alpha\hat{j})\cdot(\cos\beta\hat{i} + \sin\beta\hat{j}) = (1)(1)\cos(\alpha+\beta)$
- $\implies \cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha+\beta)$

- On the other hand,
  - $\hat{a}\times\hat{b} = |\hat{a}||\hat{b}|\sin(\alpha+\beta)\hat{k}$
  - $\implies (\cos\alpha\hat{i} - \sin\alpha\hat{j})\times(\cos\beta\hat{i} + \sin\beta\hat{j}) = (1)(1)\sin(\alpha+\beta)\hat{k}$
  - $\implies \cos\alpha\sin\beta\hat{k} + \sin\alpha\sin\beta\hat{k} = \sin(\alpha+\beta)\hat{k}$
  - $\implies \cos\alpha\sin\beta + \sin\alpha\sin\beta = \sin(\alpha+\beta)$

- Exercise 6.1(9) Prove the following by vector method,
  - (i) $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
  - (ii) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
  - 

படம் 6.10

- $\hat{a} = \cos\alpha\hat{i} + \sin\alpha\hat{j}$ and $\hat{b} = \cos\beta\hat{i} + \sin\beta\hat{j}$
  - $\hat{b}, \hat{a}, \hat{k}$ form a right-handed system.
  - $\hat{b} \times \hat{a} = |\hat{b}||\hat{a}|\sin(\alpha - \beta)\hat{k} = \sin(\alpha - \beta)\hat{k}$
  - $\hat{b} \times \hat{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\hat{k}$
- On the other hand,
  - $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\alpha - \beta) = \cos(\alpha - \beta)$
  - But $\vec{a} \cdot \vec{b} = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.

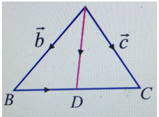
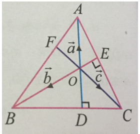
- Example 6.6 (Apollonius's theorem)
  - If D is the midpoint of the side BC of a triangle ABC, then show by vector method that $|\vec{AB}|^2 + |\vec{AC}|^2 = 2(|\vec{AD}|^2 + |\vec{BD}|^2)$.
- 

Fig. 6.11
- $\vec{AB} = \vec{b}, \vec{AC} = \vec{c} \implies \vec{AD} = \frac{1}{2}(\vec{b} + \vec{c}), \vec{BD} = \frac{1}{2}(\vec{c} - \vec{b})$
- $|\vec{AD}|^2 = \vec{AD} \cdot \vec{AD} = \frac{1}{4}(|\vec{b}|^2 + |\vec{c}|^2 + 2\vec{b} \cdot \vec{c})$
- $|\vec{BD}|^2 = \vec{BD} \cdot \vec{BD} = \frac{1}{4}(|\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b} \cdot \vec{c})$
- $\therefore |\vec{AD}|^2 + |\vec{BD}|^2 = \frac{1}{4}(2|\vec{b}|^2 + 2|\vec{c}|^2)$
- $\therefore |\vec{AD}|^2 + |\vec{BD}|^2 = \frac{1}{2}(|\vec{AB}|^2 + |\vec{AC}|^2)$
- $\therefore 2(|\vec{AD}|^2 + |\vec{BD}|^2) = |\vec{AB}|^2 + |\vec{AC}|^2$

- To prove that the altitudes of a triangle are concurrent.
- 

படம் 6.12
- AD, BE and CF are the altitudes. Suppose AD and BE meet at O.
- $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}, \vec{OC} = \vec{c}$. Then $\vec{AB} = \vec{b} - \vec{a}, \vec{BC} = \vec{c} - \vec{b}, \vec{CA} = \vec{c} - \vec{a}$
- $\vec{OA}$ lies on $\vec{AD}$, and $\vec{OB}$ lies on $\vec{BE}$.
- $\therefore \vec{OA} \cdot \vec{BC} = 0$ and $\vec{OB} \cdot \vec{CA} = 0$.
- $\Longrightarrow \vec{a} \cdot (\vec{c} - \vec{b}) = 0$ and $\vec{b} \cdot (\vec{a} - \vec{c}) = 0$
- $\Longrightarrow \vec{c} \cdot (\vec{b} - \vec{a}) = 0 \Longrightarrow \vec{OC} \cdot \vec{AB} = 0 \Longrightarrow \vec{OC}$ lies on CF.

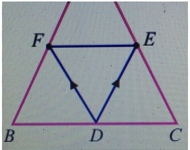
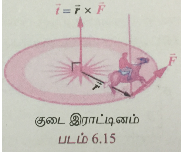
- Example 6.8: In triangle ABC, the points D, E, F, are the midpoints of the sides $BCCA$, and AB respectively. Using vector method, show that the area of ΔDEF is equal to $\frac{1}{4}$(area of ΔABC) .
- 

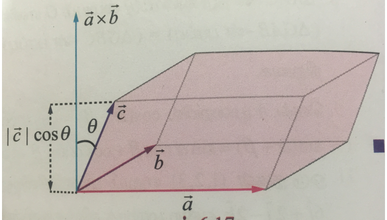
Fig. 6.13
- $\vec{AB} = \vec{b}, \vec{AC} = \vec{c} \Longrightarrow \vec{AD} = \frac{1}{2}(\vec{b} + \vec{c}), \vec{AF} = \frac{1}{2}\vec{b}, \vec{AE} = \frac{1}{2}\vec{c}.$
- Area of $\Delta\ ABC = \frac{1}{2}|\vec{b} \times \vec{c}|$
- Area of $\Delta\ DEF = \frac{1}{2}|\vec{DE} \times \vec{DF}| = \frac{1}{2}|(\vec{AE} - \vec{AD}) \times (\vec{AF} - \vec{AD})|$
- $= \frac{1}{2}\left|\left(\frac{1}{2}\vec{c} - \frac{1}{2}(\vec{b} + \vec{c})\right) \times \left(\frac{1}{2}\vec{b} - \frac{1}{2}(\vec{b} + \vec{c})\right)\right|$
- $= \frac{1}{2}\left|\left(-\frac{1}{2}\vec{b}\right) \times \left(-\frac{1}{2}\vec{c}\right)\right| = \frac{1}{4}\left|\frac{1}{2}(\vec{b} \times \vec{c})\right| = \frac{1}{4}$ (Area of $\Delta\ ABC$)

- 

குடை இராட்டினம்
படம் 6.15

- If a force $\vec{F}$ is applied on a particle at a point with position vector $\vec{r}$, then the torque or moment on the particle is given by $\vec{t} = \vec{r} \times \vec{F}$. The torque is also called the rotational force.

Applications of Vectors to geometry

- 

படம் 6.17
- Volume of the parallelepiped $=$ Area of the base $\times$ height
- Area of the base $= |\vec{a} \times \vec{b}|$
- Height $= |\vec{c}|\cos\theta$, where θ is the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$
- $\therefore$ Volume of the parallelepiped $= |\vec{a} \times \vec{b}| \times |\vec{c}|\cos\theta = (\vec{a} \times \vec{b}) \cdot \vec{c}$

Applications of Vectors:Scalar Triple Products

- Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.
- Then $(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1\hat{i} + c2\hat{j} + c_3\hat{k})$
- $= \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$
- $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$
- $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$
- $[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$
- $[\vec{a}, \vec{a}, \vec{c}] = 0 = [\vec{a}, \vec{b}, \vec{b}] = [\vec{a}, \vec{b}, \vec{a}]$
- $[\vec{a}, \vec{b}, \vec{c}] = -[\vec{b}, \vec{a}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}] = [\vec{c}, \vec{b}, \vec{a}]$
- $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\iff [\vec{a}, \vec{b}, \vec{c}] = 0 \iff \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0$

- Exercise 6.2 (9): If the vectors $a\hat{i}+a\hat{j}+c\hat{k}, \hat{i}+\hat{k}$, and $c\hat{i}+c\hat{j}+b\hat{k}$ are coplanar, prove that c is the geometric mean of a and b.
  - Coplanar $\implies \begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$
  - $\implies a(0-c)-a(b-c)+c(c-0)=0 \implies c^2=ab.$
- Exercise 6.2 (10):Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non-zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. If the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, show that $[\vec{a}, \vec{b}, \vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$.
  - $\vec{c} = \frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} \implies [\vec{a}, \vec{b}, \vec{c}]^2 = \left\{(\vec{a}\times\vec{b})\cdot\vec{c}\right\}^2$
  - $\implies [\vec{a}, \vec{b}, \vec{c}]^2 = \left\{(\vec{a}\times\vec{b})\cdot\left(\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}\right)\right\}^2 = \left\{\frac{|(\vec{a}\times\vec{b})|^2}{|\vec{a}\times\vec{b}|}\right\}^2 = |(\vec{a}\times\vec{b})|^2$
  - $= |\vec{a}|^2|\vec{b}|^2\sin^2(\pi/6) = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2.$

- Example 6.18: Prove that $[\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}+\vec{c}] = [\vec{a}, \vec{b}, \vec{c}]$
- Solution:
  $[\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}+\vec{c}] = [\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{a}+\vec{b}] + [\vec{a}+\vec{c}, \vec{a}+\vec{b}, \vec{c}]$
- $= 0 + [\vec{a}, \vec{a}+\vec{b}, \vec{c}] + [\vec{c}, \vec{a}+\vec{b}, \vec{c}]$
- $= [\vec{a}, \vec{a}, \vec{c}] + [\vec{a}, \vec{b}, \vec{c}] + 0$
- $= [\vec{a}, \vec{b}, \vec{c}]$

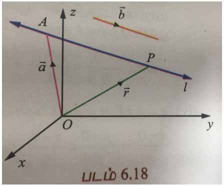
Applications of vectors: Vector Triple Products

- For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, the following are called vector triple products:
  - $(\vec{a} \times \vec{b}) \times \vec{c}$, $\vec{a} \times (\vec{b} \times \vec{c})$, $(\vec{b} \times \vec{c}) \times \vec{a}$, $\vec{b} \times (\vec{c} \times \vec{a})$,
  - $(\vec{c} \times \vec{a}) \times \vec{b}$, $\vec{c} \times (\vec{a} \times \vec{b})$, $(\vec{b} \times \vec{a}) \times \vec{c}$, $\vec{b} \times (\vec{a} \times \vec{c})$,
  - $(\vec{c} \times \vec{b}) \times \vec{a}$, $\vec{c} \times (\vec{b} \times \vec{a})$, $(\vec{a} \times \vec{c}) \times \vec{b}$, $\vec{a} \times (\vec{c} \times \vec{b})$
- $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$
- In $(\vec{a} \times \vec{b}) \times \vec{c}$, consider the vectors inside the parentheses, call $\vec{b}$ as the middle vector and $\vec{a}$ as the non-middle vector. Similarly, in $\vec{b} \times (\vec{a} \times \vec{c})$, $\vec{b}$ is the middle vector and $\vec{c}$ is the non-middle vector.
- Then we observe that, given a set of three vectors,
  - a vector triple product of these vectors is equal to
    - λ (middle vector) $-$ μ (non-middle vector)
- where λ is the dot product of the vectors other than the middle vector and μ is the dot product of the vectors other than the non-middle vector.

- Jacobi's Identity:
    - For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have
        - $(\vec{a} \times \vec{b}) \times \vec{c} + (\vec{b} \times \vec{c}) \times \vec{a} + (\vec{c} \times \vec{a}) \times \vec{b} = \vec{0}$
- Proof:
    - $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$
    - $(\vec{b} \times \vec{c}) \times \vec{a} = (\vec{a} \cdot \vec{b})\vec{c} - (\vec{a} \cdot \vec{c})\vec{b}$
    - $(\vec{c} \times \vec{a}) \times \vec{b} = (\vec{b} \cdot \vec{c})\vec{a} - (\vec{a} \cdot \vec{b})\vec{c}$
- Lagranges identity
    - For any four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$, we have
    - $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$
- Proof:
    - Put $\vec{p} = \vec{a} \times \vec{b}$.
    - $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \vec{p} \cdot (\vec{c} \times \vec{d}) = (\vec{p} \times \vec{c}) \cdot \vec{d}$
    - $= ((\vec{a} \times \vec{b}) \times \vec{c}) \cdot \vec{d}$
    - $= ((\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}) \cdot \vec{d}$
    - $= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$

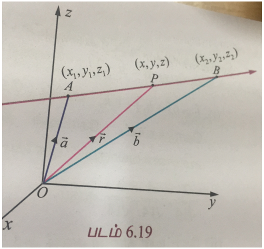
- Exercise 6.3(6):If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar vectors, then show that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{0}$.
- Solution:
  - Put $\vec{p} = \vec{a} \times \vec{b}$.
  - $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \vec{p} \times (\vec{c} \times \vec{d})$
  - $= (\vec{p} \cdot \vec{d})\vec{c} - (\vec{p} \cdot \vec{c})\vec{d} = ((\vec{a} \times \vec{b}) \cdot \vec{d})\vec{c} - ((\vec{a} \times \vec{b}) \cdot \vec{c})\vec{d}$
  - $= [\vec{a}, \vec{b}, \vec{d}]\vec{c} - [\vec{a}, \vec{b}, \vec{c}])\vec{d} = \vec{0}$
- Exercise 6.3(7): If $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}, \vec{b} = 2\hat{i} - \hat{j} + \hat{k}, \vec{c} = 3\hat{i} + 2\hat{j} + \hat{k}$, and $\vec{a} \times (\vec{b} \times \vec{c}) = l\vec{a} + m\vec{b} + n\vec{c}$, find the values of l, m, n.
- Exercise 6.3(8): Let $\hat{a}, \hat{b}, \hat{c}$ be three unit vectors and $\hat{b}, \hat{c}$ are non-parallel, and $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b}$, find the angle between $\hat{a}$ and $\hat{c}$.
- Solution. $\hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2}\hat{b} \Longrightarrow (\hat{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c} = \frac{1}{2}\hat{b}$
- $(\hat{a} \cdot \hat{c})(\hat{b} \times \vec{c}) - (\hat{a} \cdot \hat{b})(\hat{c} \times \vec{c}) = \frac{1}{2}(\hat{b} \times \vec{c})$
- $\hat{b}, \hat{c}$ are non-parallel $\Longrightarrow \hat{b} \times \hat{c} \neq \vec{0}$
- $\therefore \hat{a} \cdot \hat{c} = \frac{1}{2} \Longrightarrow \cos\theta = \frac{1}{2} \Longrightarrow \theta = \pi/3.$

Applications of Vectors to geometry:Equation of a line passing through a given point and parallel to a given direction

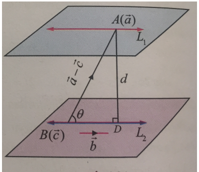
- 

படம் 6.18
- $\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{P} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \implies \vec{AP} = \vec{r} - \vec{a}$
- $\vec{AP}$ is parallel to $\vec{b} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$
- $\vec{AP} = t\vec{b}, t \in \mathbb{R} \implies \vec{r} = \vec{a} + t\vec{b}, t \in \mathbb{R}$
- $x - x_1 = t\alpha, y - y_1 = \beta, z - z_1 = t\gamma, t \in \mathbb{R}$
- $\implies \frac{x-x_1}{\alpha} = \frac{y-y_1}{\beta} = \frac{z-z_1}{\gamma} = t, t \in \mathbb{R}$

Applications of Vectors to geometry:Equation of a line passing through two given points

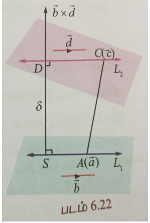
- 

- $\vec{OA} = \vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}, \vec{OB} = \vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$
- $\vec{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\vec{AP}$ is parallel to $\vec{AB}$
- $\vec{AP} = t\vec{AB}, t \in \mathbb{R} \implies \vec{r} = \vec{a} + t(\vec{b} - \vec{a}), t \in \mathbb{R}$
- $\implies \vec{r} = (1-t)\vec{a} + t\vec{b}, t \in \mathbb{R}$
- $x - x_1 = t(x_2 - x_1), y - y_1 = t(y_2 - y_1), z - z_1 = t(z_2 - z_1)$
- $\implies \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$

- 

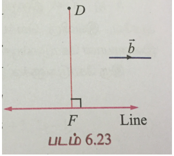
படம் 6.21
- $\vec{r} = \vec{a} + t\vec{b}, t \in \mathbb{R}$ and $\vec{r} = \vec{c} + t\vec{b}, s \in \mathbb{R}$
- $\vec{OA} = \vec{a}, \vec{OB} = \vec{c}$
- d is the distance between the parallel lines.
- $d = AD = AB \sin\theta = |\vec{c} - \vec{a}| \sin\theta$.
- $\sin\theta = \frac{|\vec{AB} \times \vec{b}|}{|\vec{AB}||\vec{b}|} = \frac{|(\vec{c}-\vec{a}) \times \vec{b}|}{|\vec{c}-\vec{a}||\vec{b}|}$.
- $d = |\vec{c} - \vec{a}| \frac{|(\vec{c}-\vec{a}) \times \vec{b}|}{|\vec{c}-\vec{a}||\vec{b}|} = \frac{|(\vec{c}-\vec{a}) \times \vec{b}|}{|\vec{b}|}$.

Applications of Vectors:Shortest distance between the two skew lines

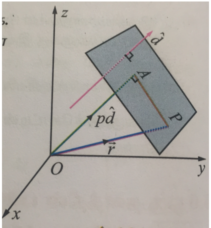
- To find the shortest distance between the two skew lines $\vec{r} = \vec{a} + s\vec{b}$ and $\vec{r} = \vec{c} + t\vec{d}$
- 

- $\vec{OA} = \vec{a}$ and $\vec{OC} = \vec{c} \implies \vec{AC} = \vec{c} - \vec{a}$.
- SD is the line segment perpendicular to both the lines.
- $\vec{SD}$ is perpendicular to both $\vec{b}$ and $\vec{d}$.
- $\therefore$ unit vector along $\vec{SD}$ is $\frac{\vec{b}\times\vec{d}}{|\vec{b}\times\vec{d}|}$
- S.D.$=|\vec{SD}|$ = the projection of $\vec{AC}$ on $\vec{SD}$ $= \left|\frac{(\vec{c}-\vec{a})\cdot(\vec{b}\times\vec{d})}{|\vec{b}\times\vec{d}|}\right|, \vec{b}\times\vec{d} \neq \vec{0}$

Applications of Vectors:Foot of the perpendicular from a given point to a straight line

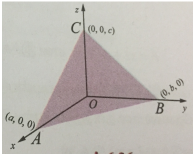
- Example 6.37: Find the coordinates of the foot of the perpendicular drawn from the point $(-1, 2, 3)$ to the straight line $\vec{r} = \hat{i} - 4\hat{j} + 3\hat{k} + t(2\hat{i} + 3\hat{j} + \hat{k})$. Also, find the shortest distance from the point to the straight line.
- 

- $\vec{OD} = -\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{OF} = \hat{i} - 4\hat{j} + 3\hat{k} + t_1(2\hat{i} + 3\hat{j} + \hat{k})$.
- $\vec{DF} = (2t_1 + 2)\hat{i} + (3t_1 - 6)\hat{j} + t_1\hat{k}$ is perpendicular to the given line which is parallel to $2\hat{i} + 3\hat{j} + \hat{k} \implies t_1 = 1$.
- $\vec{OF} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{DF} = 4\hat{i} - 3\hat{j} + \hat{k}$
- $DF = |\vec{DF}| = \sqrt{26}$

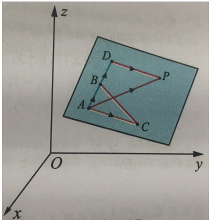
- To find the equation of the plane at a distance p from the origin and perpendicular to the unit normal vector $\hat{d}$
- 

படம் 6.24
- $\vec{OA} = p\hat{d}, \vec{AP} = \vec{r} - p\hat{d}, \vec{OA} \cdot \vec{AP} = 0 \implies \vec{r} \cdot \hat{d} = p$
- $\hat{d} = l\hat{i} + m\hat{j} + n\hat{k} \implies lx + my + nz = p.$

Applications of Vectors

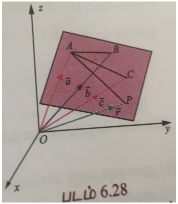
- To find the intercept form of the equation of a plane.
- 

படம் 6.26
- Assume the equation as $\vec{r} \cdot \vec{n} = p$.
- $a\hat{i}, b\hat{j}, c\hat{k}$ lie on the plane.
- $\therefore\ a\hat{i} \cdot \vec{n} = p, b\hat{j} \cdot \vec{n} = p, a\hat{k} \cdot \vec{n} = p$
- $\vec{r} \cdot \vec{n} = p \Longrightarrow (x\hat{i} + y\hat{j} + z\hat{k}).\vec{n} = p \Longrightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

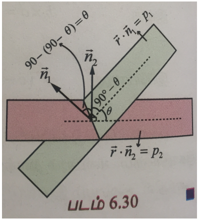
- 

படம் 6.27

Applications of Vectors

- 

Applications of Vectors

- 

Applications of Vectors

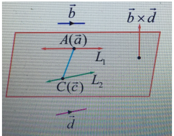
- 

Fig. 6.30

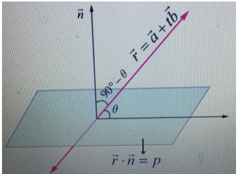
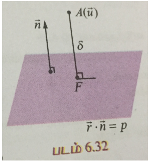
- 

Fig. 6.31

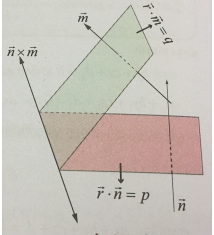
- To find the perpendicular distance from a point with position vector $\vec{u}$ to the plane $\vec{r} \cdot \vec{n} = p$



- Equation of AF is $\vec{r} = \vec{u} + t\vec{n}$
- The p.v. of F is $\vec{r_1} = \vec{u} + t_1\vec{n}$
- $\therefore (\vec{u} + t_1\vec{n}) \cdot \vec{n} = p \Longrightarrow t_1 = \frac{p - \vec{u} \cdot \vec{n}}{|\vec{n}|^2}$
- $\delta = \left|\vec{FA}\right| = |\vec{u} - (\vec{u} + t_1\vec{n})| = |-t_1\vec{n}| = \left|\frac{\vec{u} \cdot \vec{n} - p}{|\vec{n}|}\right|$

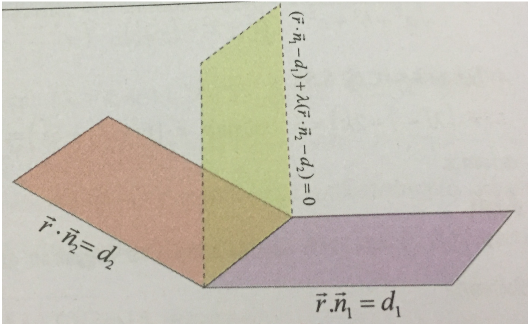
Applications of Vectors

- To find the line of intersection of the planes $\vec{r} \cdot \vec{n} = p$ and $\vec{r} \cdot \vec{m} = q$



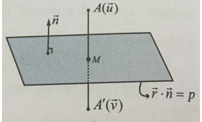
படம் 6.33

- $\vec{n} \times \vec{m}$ is a vector parallel to the required line.
- $\vec{a}$ is a point on the required line.
- $\vec{r} = \vec{a} + t\vec{n} \times \vec{m}$.

- To find the vector equation of a plane which passes through the line of intersection of the planes $\vec{r}\cdot\vec{n_1} = d_1$ and $\vec{r}\cdot\vec{n_2} = d_2$
- 

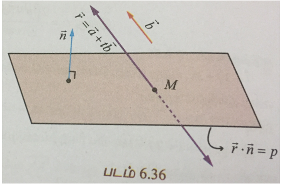
படம் 6.34
- $(\vec{r}\cdot\vec{n_1} - d_1) + \lambda(\vec{r}\cdot\vec{n_2} - d_2) = 0$ where $\lambda \in \mathbb{R}$.
- $\vec{r}\cdot(\vec{n_1} + \lambda\vec{n_2}) - (d_1 + \lambda d_2) = 0$ where $\lambda \in \mathbb{R}$.

Applications of Vectors

- To find the image of the point $\vec{u}$ in the plane $\vec{r} \cdot \vec{n} = p$
- 

படம் 6.35
- $\vec{v} = \vec{u} + \frac{2[p - \vec{u} \cdot \vec{n}]}{|\vec{n}|^2}$

Applications of Vectors

- To find the position vector of the point of intersection of the straight line $\vec{r} = \vec{a} + t\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = p$, where $\vec{b} \cdot \vec{n} \neq 0$.
- 

- $\vec{r_1} = \vec{a} + \left(\frac{p - (\vec{a} \cdot \vec{n})}{\vec{b} \cdot \vec{n}} \right) \vec{b}$